COMPARISON THEOREMS
FOR THE SMALL BALL PROBABILITIES
OF GAUSSIAN PROCESSES IN WEIGHTED L_2-NORMS

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Dedicated to Boris Mikhailovich Makarov, with great respect

ABSTRACT. We prove comparison theorems for small ball probabilities of the Green Gaussian processes in weighted L_2-norms. We find the sharp small ball asymptotics for many classical processes under quite general assumptions on the weight.

1. INTRODUCTION

The problem of small ball behavior of a random process X in the norm $\|\cdot\|$ is to describe the asymptotics as $\varepsilon \to 0$ of the probability $\mathsf{P}\{\|X\| \leq \varepsilon\}$. The theory of small ball behavior for Gaussian processes in various norms is intensively developed in recent decades, see surveys [16], [15] and the site [17].

Suppose we have a Gaussian process $X(t)$, $0 \leqslant t \leqslant 1$, with zero mean and covariance function $G_X(t,s) = \mathsf{E}X(t)X(s)$, $t, s \in [0,1]$. Let ψ be a non-negative weight function on $[0,1]$. We set

$$\|X\|_\psi = \left(\int_0^1 X^2(t)\psi(t)dt\right)^{\frac{1}{2}}$$

(we drop the subscript ψ if $\psi \equiv 1$).

By the classical Karhunen–Loève expansion, one has the equality in distribution

$$\|X\|_\psi^2 = \sum_{j=1}^\infty \lambda_j \xi_j^2.$$

Here ξ_j, $j \in \mathbb{N}$, are independent standard Gaussian random variables while $\lambda_j > 0$, $j \in \mathbb{N}$, are the eigenvalues of the integral equation

$$\lambda f(t) = \int_0^1 G(t,s)\sqrt{\psi(t)\psi(s)}f(s)\,ds, \quad t \in [0;1].$$

In the papers [19, 20] there was selected the concept of the *Green* process, i.e. Gaussian process with covariance being the Green function for a self-adjoint differential operator. The approach developed in these papers allows to obtain the sharp (up to a constant) asymptotics of small deviations in L_2-norm for this class of processes. In the papers [1, 2], using this approach, we have calculated the sharp asymptotics of small ball probabilities for a large class of particular processes with various weights.

In this paper we prove a comparison theorem for the small ball probabilities of the Green Gaussian processes in the weighted L_2-norms. This theorem gives us the opportunity to obtain the sharp small ball asymptotics for many classical processes under quite general assumptions on the weight. For the Wiener process and some other processes this result was obtained in [4].

Let us recall some notation. A function $G(t,s)$ is called the Green function of a boundary value problem for differential operator L if it satisfies the equation $LG = \delta(t-s)$ in the sense of distributions and satisfies the boundary conditions.

The space $W_p^m(0,1)$ is the Banach space of functions y having continuous derivatives up to $(m-1)$-th order when $y^{(m-1)}$ is absolutely continuous on $[0,1]$ and $y^{(m)} \in L_p(0,1)$.

$\mathcal{V}(\dots)$ stands for the Vandermonde determinant.

2. The calculation of the perturbation determinant

Let L be a self-adjoint differential operator of order $2n$, generated by the differential expression

$$Lv \equiv (-1)^n v^{(2n)} + \left(p_{n-1} v^{(n-1)}\right)^{(n-1)} + \dots + p_0 v; \tag{1}$$

and boundary conditions

$$U_\nu(v) \equiv U_{\nu 0}(v) + U_{\nu 1}(v) = 0, \quad \nu = 1, \dots, 2n. \tag{2}$$

Here

$$U_{\nu 0}(v) = \alpha_\nu v^{(k_\nu)}(0) + \sum_{j=0}^{k_\nu - 1} \alpha_{\nu j} v^{(j)}(0),$$

$$U_{\nu 1}(v) = \gamma_\nu v^{(k_\nu)}(1) + \sum_{j=0}^{k_\nu - 1} \gamma_{\nu j} v^{(j)}(1),$$

and for any ν at least one of coefficients α_ν and γ_ν is not zero.

We assume that the system of boundary conditions (2) is normalized. This means that the sum of orders of all boundary conditions $\varkappa = \sum_\nu k_\nu$ is minimal. See [3, §4]; see also [10] where a more general class of boundary value problems is considered.

We introduce the notation

$$\widetilde{\alpha}_\nu = \alpha_\nu (\psi(0))^{\frac{k_\nu}{2n} - \frac{2n-1}{4n}}, \quad \widetilde{\gamma}_\nu = \gamma_\nu (\psi(1))^{\frac{k_\nu}{2n} - \frac{2n-1}{4n}}, \quad \omega_k = \exp(ik\pi/n),$$

$$\theta_1(\psi) = \det \begin{pmatrix} \widetilde{\gamma}_1 & \widetilde{\alpha}_1 \omega_1^{k_1} & \dots & \widetilde{\alpha}_1 \omega_{n-1}^{k_1} & \widetilde{\alpha}_1 \omega_n^{k_1} & \widetilde{\gamma}_1 \omega_{n+1}^{k_1} & \dots & \widetilde{\gamma}_1 \omega_{2n-1}^{k_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{\gamma}_{2n} & \widetilde{\alpha}_{2n} \omega_1^{k_{2n}} & \dots & \widetilde{\alpha}_{2n} \omega_{n-1}^{k_{2n}} & \widetilde{\alpha}_{2n} \omega_n^{k_{2n}} & \widetilde{\gamma}_{2n} \omega_{n+1}^{k_{2n}} & \dots & \widetilde{\gamma}_{2n} \omega_{2n-1}^{k_{2n}} \end{pmatrix},$$

$$\theta_{-1}(\psi) = \det \begin{pmatrix} \widetilde{\alpha}_1 & \widetilde{\alpha}_1 \omega_1^{k_1} & \dots & \widetilde{\alpha}_1 \omega_{n-1}^{k_1} & \widetilde{\gamma}_1 \omega_n^{k_1} & \widetilde{\gamma}_1 \omega_{n+1}^{k_1} & \dots & \widetilde{\gamma}_1 \omega_{2n-1}^{k_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{\alpha}_{2n} & \widetilde{\alpha}_{2n} \omega_1^{k_{2n}} & \dots & \widetilde{\alpha}_{2n} \omega_{n-1}^{k_{2n}} & \widetilde{\gamma}_{2n} \omega_n^{k_{2n}} & \widetilde{\gamma}_{2n} \omega_{n+1}^{k_{2n}} & \dots & \widetilde{\gamma}_{2n} \omega_{2n-1}^{k_{2n}} \end{pmatrix}.$$

Theorem 1. *Let L be a self-adjoint differential operator of order $2n$, generated by the differential expression (1) and boundary conditions (2). Let also $p_m \in W_\infty^m(0,1)$, $m = 0, \dots, n-1$.*

Consider two eigenvalue problems

$$Ly = \mu\psi_{1,2}y; \qquad U_\nu(y) = 0, \quad \nu = 1, \dots, 2n, \tag{3}$$

where $\psi_1, \psi_2 \in W_\infty^n(0,1)$. Suppose that the weight functions ψ_1, ψ_2 are bounded away from zero, and

$$\int_0^1 \psi_1^{\frac{1}{2n}}(x)dx = \int_0^1 \psi_2^{\frac{1}{2n}}(x)dx = \vartheta. \tag{4}$$

Denote by $\mu_k^{(j)}$, $j = 1,2$, $k \in \mathbb{N}$, the eigenvalues of the problems (3), enumerated in ascending order according to the multiplicity. Then

$$\prod_{k=1}^\infty \frac{\mu_k^{(1)}}{\mu_k^{(2)}} = \left| \frac{\theta_{-1}(\psi_2)}{\theta_{-1}(\psi_1)} \right|.$$

Proof. Consider the first problem in (3). Denote by $\varphi_j(t, \zeta)$, $j = 0, \ldots, 2n - 1$, solutions of the equation $Ly = \zeta^{2n}\psi_1 y$, specified by the initial conditions $\varphi_j^{(k)}(0, \zeta) = \delta_{jk}$.

We substitute a general solution of the equation $y(t) = c_0\varphi_0(t, \zeta) + \ldots + c_{2n-1}\varphi_{2n-1}(t, \zeta)$ to the boundary conditions and obtain $\mu_k^{(1)} = x_k^{2n}$, where $x_1 \leq x_2 \leq \ldots$ are positive roots of the function

$$F_1(\zeta) = \det \begin{pmatrix} U_1(\varphi_0) & \cdots & U_1(\varphi_{2n-1}) \\ \vdots & \ddots & \vdots \\ U_{2n}(\varphi_0) & \cdots & U_{2n}(\varphi_{2n-1}) \end{pmatrix}.$$

It is easy to see ([3, §2]) that $F_1(\zeta)$ is an entire function.

It is well known (see [9], [3, §4]) that there exist solutions $\widetilde{\varphi}_j(t, \zeta)$, $j = 0, \ldots, 2n - 1$, of the equation $Ly = \zeta^{2n}\psi_1 y$ such that for large $|\zeta|$, $|\arg(\zeta)| \leq \frac{\pi}{2n}$, the following asymptotic relation holds:

$$\widetilde{\varphi}_j(t, \zeta) = (\psi_1(t))^{-\frac{2n-1}{4n}} \exp\left(i\omega_j\zeta \int_0^t \psi_1^{\frac{1}{2n}}(u)du\right)\left(1 + O(|\zeta|^{-1})\right), \quad j = 0, \ldots, 2n - 1. \quad (5)$$

The relation (5) is uniform in $t \in [0, 1]$, and one can differentiate it.

It is easy to see that for $|\arg(\zeta)| \leq \frac{\pi}{2n}$, $|\zeta| \to \infty$

$$U_\nu(\widetilde{\varphi}_j) = \left(\alpha_\nu\widetilde{\varphi}_j^{(k_\nu)}(0, \zeta) + \gamma_\nu\widetilde{\varphi}_j^{(k_\nu)}(1, \zeta)\right)\left(1 + O(|\zeta|^{-1})\right).$$

For large $|\zeta|$, the functions $\widetilde{\varphi}_j(t, \zeta)$ are linearly independent. Therefore there exists a matrix $C(\zeta) = (c_{jk})_{0 \leq j,k \leq 2n-1}$ depending on ζ such that

$$(\varphi_0(t, \zeta), \ldots, \varphi_{2n-1}(t, \zeta))^\top = C(\zeta)(\widetilde{\varphi}_0(t, \zeta), \ldots, \widetilde{\varphi}_{2n-1}(t, \zeta))^\top.$$

Thus,

$$F_1(\zeta) = \det(C(\zeta)) \cdot \det \begin{pmatrix} U_1(\widetilde{\varphi}_0) & \cdots & U_{2n}(\widetilde{\varphi}_0) \\ \vdots & \ddots & \vdots \\ U_1(\widetilde{\varphi}_{2n-1}) & \cdots & U_{2n}(\widetilde{\varphi}_{2n-1}) \end{pmatrix}. \quad (6)$$

By the initial conditions we have

$$I_{2n} = C(\zeta) \begin{pmatrix} \widetilde{\varphi}_0(0, \zeta) & \cdots & \widetilde{\varphi}_0^{(2n-1)}(0, \zeta) \\ \vdots & \ddots & \vdots \\ \widetilde{\varphi}_{2n-1}(0, \zeta) & \cdots & \widetilde{\varphi}_{2n-1}^{(2n-1)}(0, \zeta) \end{pmatrix}.$$

By the relations (5), we obtain for $|\arg(\zeta)| \leq \frac{\pi}{2n}$, $|\zeta| \to \infty$

$$\det \begin{pmatrix} \widetilde{\varphi}_0(0, \zeta) & \cdots & \widetilde{\varphi}_0^{(2n-1)}(0, \zeta) \\ \vdots & \ddots & \vdots \\ \widetilde{\varphi}_{2n-1}(0, \zeta) & \cdots & \widetilde{\varphi}_{2n-1}^{(2n-1)}(0, \zeta) \end{pmatrix} = (\psi_1(0))^{2n\left(-\frac{1}{2}+\frac{1}{4n}\right)} \times$$

$$\times \det \begin{pmatrix} 1 & \left(i\zeta(\psi_1(0))^{\frac{1}{2n}}\right)^1 & \cdots & \left(i\zeta(\psi_1(0))^{\frac{1}{2n}}\right)^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \left(i\omega_{2n-1}\zeta(\psi_1(0))^{\frac{1}{2n}}\right)^1 & \cdots & \left(i\omega_{2n-1}\zeta(\psi_1(0))^{\frac{1}{2n}}\right)^{2n-1} \end{pmatrix}(1 + O(|\zeta|^{-1})) =$$

$$= (\psi_1(0))^{\frac{1-2n}{2}} \left(i\zeta(\psi_1(0))^{\frac{1}{2n}}\right)^{n(2n-1)} \mathcal{V}(1, \omega_1, \ldots, \omega_{2n-1})(1 + O(|\zeta|^{-1})) =$$

$$= (i\zeta)^{2n^2-n} \mathcal{V}(1, \omega_1, \ldots, \omega_{2n-1})(1 + O(|\zeta|^{-1})).$$

Whence, for $|\arg(\zeta)| \leq \frac{\pi}{2n}$, $|\zeta| \to \infty$, we have

$$\det(C(\zeta)) = \frac{(i\zeta)^{n-2n^2}}{\mathcal{V}(1, \omega_1, \ldots, \omega_{2n-1})} \cdot (1 + O(|\zeta|^{-1})).$$

Next, following [3, §4], we obtain for $|\arg(\zeta)| \leqslant \frac{\pi}{2n}$, $|\zeta| \to \infty$

$$\det \begin{pmatrix} U_1(\widetilde{\varphi}_0) & \dots & U_1(\widetilde{\varphi}_{2n-1}) \\ \vdots & \ddots & \vdots \\ U_{2n}(\widetilde{\varphi}_0) & \dots & U_{2n}(\widetilde{\varphi}_{2n-1}) \end{pmatrix} =$$

$$= (i\zeta)^{\varkappa} \exp\left(-i\omega_1 \vartheta\zeta - i\omega_2 \vartheta\zeta - \dots - i\omega_{n-1}\vartheta\zeta\right) \times$$
$$\times \left(\theta_1(\psi_1)\exp(i\vartheta\zeta) + \theta_0(\psi_1) + \theta_{-1}(\psi_1)\exp(-i\vartheta\zeta)\right)\left(1 + O(|\zeta|^{-1})\right)$$

(we recall that $\varkappa = k_1 + \dots + k_{2n}$), where $\theta_0(\psi_1)$ is some unimportant constant.

It is easy to see ([19, Theorem 1.1]) that $\theta_1(\psi_1) = -\omega_1^{\varkappa}\theta_{-1}(\psi_1)$.

Substituting these formulas to (6) we obtain for $|\arg(\zeta)| \leqslant \frac{\pi}{2n}$, $|\zeta| \to \infty$

$$F_1(\zeta) = \frac{(i\zeta)^{n-2n^2+\varkappa} \exp\left(-i\omega_1\vartheta\zeta - i\omega_2\vartheta\zeta - \dots - i\omega_{n-1}\vartheta\zeta\right)}{\mathcal{V}(1, \omega_1, \dots, \omega_{2n-1})} \times$$
$$\times \left(\theta_{-1}(\psi_1)(\exp(-i\vartheta\zeta) - \omega_1^{\varkappa}\exp(i\vartheta\zeta)) + \theta_0(\psi_1)\right)\left(1 + O(|\zeta|^{-1})\right).$$

Now we consider the second problem in (3) and define the function $F_2(\zeta)$ similarly to $F_1(\zeta)$ with ψ_2 instead of ψ_1. Then the following relation holds:

$$F_2(\zeta) = \frac{(i\zeta)^{n-2n^2+\varkappa} \exp\left(-i\omega_1\vartheta\zeta - i\omega_2\vartheta\zeta - \dots - i\omega_{n-1}\vartheta\zeta\right)}{\mathcal{V}(1, \omega_1, \dots, \omega_{2n-1})} \times$$
$$\times \left(\theta_{-1}(\psi_2)(\exp(-i\vartheta\zeta) - \omega_1^{\varkappa}\exp(i\vartheta\zeta)) + \theta_0(\psi_2)\right)\left(1 + O(|\zeta|^{-1})\right).$$

Whence, for $|\zeta| \to \infty$, $\arg(\zeta) \neq \frac{\pi j}{n}$, $j \in \mathbb{Z}$, we obtain

$$\left|\frac{F_2(\zeta)}{F_1(\zeta)}\right| \to \left|\frac{\theta_{-1}(\psi_2)}{\theta_{-1}(\psi_1)}\right|.$$

Moreover, the quotient $|F_2(\zeta)/F_1(\zeta)|$ is uniformly bounded on circles $|\zeta| = r_k$ for a proper sequence $r_k \to \infty$.

Further, by continuity of solutions to a differential equation with respect to parameters, we have $F_1(\zeta)/F_2(\zeta) \rightrightarrows 1$ as $\zeta \to 0$.

Applying the Jensen Theorem to $F_1(\zeta)$ and $F_2(\zeta)$, we obtain

$$\prod_{k=1}^{\infty} \frac{\mu_k^{(1)}}{\mu_k^{(2)}} = \exp\left(\lim_{\rho \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \ln \frac{|F_2(\rho e^{i\theta})|}{|F_1(\rho e^{i\theta})|} d\theta\right) = \left|\frac{\theta_{-1}(\psi_2)}{\theta_{-1}(\psi_1)}\right|.$$

$\square$

Corollary 1. *Let the covariance of a centered Gaussian process $X(t)$, $0 \leqslant t \leqslant 1$, be the Green function of a self-adjoint operator L generated by the differential expression (1) and boundary conditions (2).*

Let the coefficients p_m, $m = 0, \dots, n-1$, and the weight functions ψ_1, ψ_2 satisfy the assumptions of Theorem 1. Then

$$\lim_{\varepsilon \to 0} \frac{\mathsf{P}(\|X\|_{\psi_1} \leq \varepsilon)}{\mathsf{P}(\|X\|_{\psi_2} \leq \varepsilon)} = \left|\frac{\theta_{-1}(\psi_2)}{\theta_{-1}(\psi_1)}\right|^{1/2}.$$

Proof. Denote by $\mu_k^{(1,2)}$ the eigenvalues of the problems (3).

Using the Li comparison theorem (see [12, 14]) and Theorem 1, we obtain

$$\lim_{\varepsilon \to 0} \frac{\mathsf{P}(\|X\|_{\psi_1} \leq \varepsilon)}{\mathsf{P}(\|X\|_{\psi_2} \leq \varepsilon)} = \left(\prod_{k=1}^{\infty} \frac{\mu_k^{(1)}}{\mu_k^{(2)}}\right)^{\frac{1}{2}} = \left|\frac{\theta_{-1}(\psi_2)}{\theta_{-1}(\psi_1)}\right|^{\frac{1}{2}}.$$

$\square$

Remark. If the assumption (4) does not hold then the probabilities $\mathsf{P}(\|X\|_{\psi_{1,2}} \le \varepsilon)$ have different logarithmic asymptotics (see [20, Theorem 7.3]).

3. Separated boundary conditions

Now we consider an important particular case.

Theorem 2. *Let the assumptions of Corollary 1 be satisfied. Suppose also that the boundary conditions (2) are separated in main terms, i.e. have the form*

$$
\left.\begin{aligned}
v^{(k_\nu)}(0) + \sum_{j=0}^{k_\nu - 1} \left(\alpha_{\nu j} v^{(j)}(0) + \gamma_{\nu j} v^{(j)}(1)\right) = 0, \\
v^{(k'_\nu)}(1) + \sum_{j=0}^{k'_\nu - 1} \left(\alpha'_{\nu j} v^{(j)}(0) + \gamma'_{\nu j} v^{(j)}(1)\right) = 0,
\end{aligned}\right\}
\quad \nu = 1, \dots, n.
$$

Denote by $\varkappa_0$ and $\varkappa_1$ sums of orders of boundary conditions at zero and one, respectively: $\varkappa_0 = k_1 + \ldots + k_n$, $\varkappa_1 = k'_1 + \ldots + k'_n$. Then

$$
\lim_{\varepsilon \to 0} \frac{\mathsf{P}(\|X\|_{\psi_1} \le \varepsilon)}{\mathsf{P}(\|X\|_{\psi_2} \le \varepsilon)} = \left(\frac{\psi_2(0)}{\psi_1(0)}\right)^{-\frac{n}{4} + \frac{1}{8} + \frac{\varkappa_0}{4n}} \left(\frac{\psi_2(1)}{\psi_1(1)}\right)^{-\frac{n}{4} + \frac{1}{8} + \frac{\varkappa_1}{4n}}. \tag{7}
$$

Proof. Under assumptions of the Theorem the matrix determining $\theta_{-1}(\psi)$ is block diagonal, and we obtain

$$
\theta_{-1}(\psi) = (-1)^{\varkappa_1} (\psi(0))^{\frac{\varkappa_0}{2n} - \frac{2n-1}{4}} (\psi(1))^{\frac{\varkappa_1}{2n} - \frac{2n-1}{4}} \cdot \mathcal{V}(\omega_1^{k_1}, \dots, \omega_1^{k_n}) \cdot \mathcal{V}(\omega_1^{k'_1}, \dots, \omega_1^{k'_n}).
$$

Therefore,

$$
\frac{\theta_{-1}(\psi_2)}{\theta_{-1}(\psi_1)} = \left(\frac{\psi_2(0)}{\psi_1(0)}\right)^{\frac{\varkappa_0}{2n} - \frac{2n-1}{4}} \left(\frac{\psi_2(1)}{\psi_1(1)}\right)^{\frac{\varkappa_1}{2n} - \frac{2n-1}{4}}.
$$

$\square$

Many classical Gaussian processes satisfy the assumptions of Theorem 2. We give several examples.

For a random process $X(t)$, $0 \le t \le 1$, denote by $X_m^{[\beta_1, \dots, \beta_m]}(t)$, $0 \le t \le 1$, the m-times integrated process:

$$
X_m^{[\beta_1, \dots, \beta_m]}(t) = (-1)^{\beta_1 + \dots + \beta_m} \int_{\beta_m}^t \dots \int_{\beta_1}^{t_1} X(s) ds dt_1 \dots
$$

(any index β_ν equals 0 or 1).

Following [1], we introduce the notation

$$
z_n = \exp(i\pi/n), \quad \varepsilon_n = \left(\varepsilon \sqrt{2n \sin \frac{\pi}{2n}}\right)^{\frac{1}{2n-1}}, \quad \mathcal{D}_n = \frac{2n-1}{2n \sin \frac{\pi}{2n}}.
$$

Proposition 1. *Suppose that the function $\psi \in W_\infty^{m+1}(0,1)$ is bounded away from zero and satisfies the relation $\int_0^1 \psi^{\frac{1}{2(m+1)}}(x) dx = 1$. Then for the integrated Brownian motion the following relation holds:*

$$
\mathsf{P}(\|W_m^{[\beta_1, \dots, \beta_m]}\|_\psi \le \varepsilon) \sim \left(\frac{\psi(1)}{\psi(0)}\right)^{-\frac{m+1}{8} + \frac{\mathcal{K}}{4(m+1)}} \times
$$

$$
\times \frac{(2m+2)^{\frac{m}{2} + 1}}{|\mathcal{V}(1, z_{m+1}^{1 - 3\beta_1}, z_{m+1}^{2 - 5\beta_2}, \dots, z_{m+1}^{m + \beta_m})|} \frac{\varepsilon_{m+1}}{\sqrt{\pi \mathcal{D}_{m+1}}} \exp\left(-\frac{\mathcal{D}_{m+1}}{2 \varepsilon_{m+1}^2}\right),
$$

where $\mathcal{K} = \mathcal{K}(\beta_1, \dots, \beta_m) = \sum_{\nu=1}^m (2\nu + 1)\beta_\nu$.

5

Proof. The boundary value problem corresponding to W_m was derived in [11], see also [20]. Namely in Theorem 2 one should set $n = m + 1$,

$$k_\nu = \begin{cases} m - \nu & \text{for } \beta_\nu = 0, \\ m + 1 + \nu & \text{for } \beta_\nu = 1, \end{cases} \quad \nu = 1, \ldots, m; \quad k_{m+1} = m,$$

$$k'_\nu = 2m + 1 - k_\nu, \quad \nu = 1, \ldots, m + 1.$$

This implies $\varkappa_0 = \mathcal{K} + \frac{m(m+1)}{2}$, $\varkappa_1 = \frac{(m+1)(3m+2)}{2} - \mathcal{K}$.

We substitute these quantities into (7) and obtain

$$\mathsf{P}(\|W_m^{[\beta_1,\ldots,\beta_m]}\|_\psi \le \varepsilon) \sim \left(\frac{\psi(1)}{\psi(0)}\right)^{-\frac{m+1}{8} + \frac{\mathcal{K}}{4(m+1)}} \cdot \mathsf{P}(\|W_m^{[\beta_1,\ldots,\beta_m]}\|_\psi \le \varepsilon), \quad \varepsilon \to 0.$$

The asymptotics of probability $\mathsf{P}(\|W_m^{[\beta_1,\ldots,\beta_m]}\| \le \varepsilon)$ was obtained in [1, Proposition 1.5]. □

In a similar way, using [1, Propositions 1.6 and 1.8], we obtain the following relations.

Proposition 2. *Let $B(t)$ be the Brownian bridge. Then, under assumptions of Proposition 1, the following relation holds:*

$$\mathsf{P}(\|B_m^{[\beta_1,\ldots,\beta_m]}\|_\psi \le \varepsilon) \sim (\psi(0))^{\frac{m+1}{8} - \frac{\mathcal{K}}{4(m+1)}} (\psi(1))^{\frac{\mathcal{K}+1}{4(m+1)} - \frac{m+1}{8}} \times$$

$$\times \frac{(2m+2)^{\frac{m+1}{2}} \sqrt{2 \sin \frac{\pi}{2m+2}}}{|\mathcal{V}(z_{m+1}^{1-3\beta_1}, z_{m+1}^{2-5\beta_2}, \ldots, z_{m+1}^{m+\beta_m})|} \frac{1}{\sqrt{\pi \mathcal{D}_{m+1}}} \exp\left(-\frac{\mathcal{D}_{m+1}}{2\varepsilon_{m+1}^2}\right).$$

Proposition 3. *Let $\mathbb{B}_m(t) = (W_m(t)|W_j(1) = 0, 0 \le j \le m)$ be conditional integrated Wiener process (see [13]). Then, under assumptions of Proposition 1, the following relation holds:*

$$\mathsf{P}(\|\mathbb{B}_m\|_\psi \le \varepsilon) \sim (\psi(0)\psi(1))^{\frac{1}{8}} \times$$

$$\times \frac{(2m+2)^{\frac{m}{2}+1} \cdot \left(\prod_{j=0}^{m} \frac{j!}{(m+1+j)!}\right)^{\frac{1}{2}}}{|\mathcal{V}(1, z_{m+1}, \ldots, z_{m+1}^m)| \sqrt{\pi \mathcal{D}_{m+1}} \cdot \varepsilon_{m+1}^{m(m+2)}} \exp\left(-\frac{\mathcal{D}_{m+1}}{2\varepsilon_{m+1}^2}\right).$$

Let us introduce the notation

$$\widetilde{\varepsilon}_n = \left(\varepsilon \sqrt{n \sin \frac{\pi}{2n}}\right)^{\frac{1}{2n-1}}, \quad \widehat{\varepsilon}_n = \left(\varepsilon \sqrt{\frac{2n}{c_n} \sin \frac{\pi}{2n}}\right)^{\frac{1}{2n-1}}, \quad c_n = \frac{2\sqrt{\pi} \Gamma(n)}{\Gamma(n - \frac{1}{2})}.$$

The following relations can be obtained using [1, Theorem 2.2], [19, Theorem 2.2] and [5, Theorem 3.1].

Proposition 4. *Let $U(t)$ be the Ornstein–Uhlenbeck process, i.e. the centered Gaussian process with the covariance function $\mathsf{E}U(t)U(s) = e^{-|t-s|}$. Then, under assumptions of Proposition 1, the following relation holds:*

$$\mathsf{P}(\|U_m^{[\beta_1,\ldots,\beta_m]}\|_\psi \le \varepsilon) \sim (\psi(0))^{\frac{m+1}{8} - \frac{\mathcal{K}+1}{4(m+1)}} (\psi(1))^{\frac{\mathcal{K}}{4(m+1)} - \frac{m+1}{8}} \times$$

$$\times \frac{(2m+2)^{\frac{m+1}{2}} 2\sqrt{e} \sqrt{\sin \frac{\pi}{2m+2}}}{|\mathcal{V}(z_{m+1}^{1-3\beta_1}, z_{m+1}^{2-5\beta_2}, \ldots, z_{m+1}^{m+\beta_m})|} \frac{\widetilde{\varepsilon}_{m+1}^2}{\sqrt{\pi \mathcal{D}_{m+1}}} \exp\left(-\frac{\mathcal{D}_{m+1}}{2\widetilde{\varepsilon}_{m+1}^2}\right).$$

Proposition 5. *Let $S(t) = W(t+1) - W(t)$ be the Slepian process (see [21]). Then, under assumptions of Proposition 1, the following relation holds:*

$$\mathsf{P}(\|S_m^{[\beta_1,\ldots,\beta_m]}\|_\psi \le \varepsilon) \sim \sqrt{\frac{2}{e}} \mathsf{P}(\|U_m^{[\beta_1,\ldots,\beta_m]}\|_\psi \le \varepsilon).$$

Proposition 6. *Let $M^{(n)}(t)$ be the Matern process (see [18]), i.e. the centered Gaussian process with the covariance function*

$$\mathsf{E}M^{(n)}(t)M^{(n)}(s) = \frac{(n-1)!}{(2n-2)!}\exp(-|t-s|)\sum_{k=0}^{n-1}\frac{(n+k-1)!}{k!(n-k-1)!}(2|t-s|)^{n-k-1}.$$

Then, under assumptions of Proposition 1, the following relation holds:

$$\mathsf{P}(\|M^{(n)}\|_\psi \le \varepsilon) \sim (\psi(0)\psi(1))^{-\frac{n}{8}}\frac{\sqrt{2^{n^2+n+1}n^{n+1}e^n}}{|\mathcal{V}(1,z_n,\ldots,z_n^{n-1})|}\frac{\widehat{\varepsilon}_n^{n^2+1}}{\sqrt{\pi\mathcal{D}_n}}\exp\left(-\frac{\mathcal{D}_n}{2\widehat{\varepsilon}_n^{\,2}}\right).$$

Remark. It is well known that $\{M^{(1)}(t), 0 \le t \le 1\} \overset{law}{=} \{U(t), 0 \le t \le 1\}$. It is easy to see that the formula from Proposition 4 with $m = 0$ coincides with the formula from Proposition 6 with $n = 1$.

4. NON-SEPARATED BOUNDARY CONDITIONS

If some boundary conditions are not separated in the main terms, they can be split into pairs of the following form (see [3, §18]):

$$
\begin{aligned}
&av^{(\ell)}(0) + bv^{(\ell)}(1) + \sum_{j=0}^{\ell-1}\left(\alpha_{\nu j}v^{(j)}(0) + \gamma_{\nu j}v^{(j)}(1)\right) = 0,\\
&bv^{(2n-\ell-1)}(0) + av^{(2n-\ell-1)}(1) + \sum_{j=0}^{2n-\ell-2}\left(\alpha'_{\nu j}v^{(j)}(0) + \gamma'_{\nu j}v^{(j)}(1)\right) = 0.
\end{aligned}
\tag{8}
$$

We consider the case with a unique such pair.

Theorem 3. *Let the assumptions of Corollary 1 be satisfied. Suppose also that one pair of boundary conditions has the form (8) while other ones are separated in the main terms[1]:*

$$
\left.
\begin{aligned}
&v^{(k_\nu)}(0) + \sum_{j=0}^{k_\nu-1}\left(\alpha_{\nu j}v^{(j)}(0) + \gamma_{\nu j}v^{(j)}(1)\right) = 0,\\
&v^{(k'_\nu)}(1) + \sum_{j=0}^{k'_\nu-1}\left(\alpha_{\nu j}v^{(j)}(0) + \gamma_{\nu j}v^{(j)}(1)\right) = 0,
\end{aligned}
\right\}
\quad \nu = 1,\ldots,n-1.
$$

Denote by $\varkappa_0$ and $\varkappa_1$ the sums of orders of separated boundary conditions at zero and one, respectively: $\varkappa_0 = k_1 + \ldots + k_{n-1}$, $\varkappa_1 = k'_1 + \ldots + k'_{n-1}$. Then

$$
\lim_{\varepsilon\to 0}\frac{\mathsf{P}(\|X\|_{\psi_1} \le \varepsilon)}{\mathsf{P}(\|X\|_{\psi_2} \le \varepsilon)} = \left(\frac{\psi_2(0)}{\psi_1(0)}\right)^{\frac{\varkappa_0}{4n}-\frac{(n-1)(2n-1)}{8n}}\left(\frac{\psi_2(1)}{\psi_1(1)}\right)^{\frac{\varkappa_1}{4n}-\frac{(n-1)(2n-1)}{8n}}\times
$$

$$
\times\left|\frac{\mathcal{M}_1 a^2\left(\frac{\psi_2(1)}{\psi_2(0)}\right)^{\frac{2n-2\ell-1}{4n}} + \mathcal{M}_2 b^2\left(\frac{\psi_2(0)}{\psi_2(1)}\right)^{\frac{2n-2\ell-1}{4n}}}{\mathcal{M}_1 a^2\left(\frac{\psi_1(1)}{\psi_1(0)}\right)^{\frac{2n-2\ell-1}{4n}} + \mathcal{M}_2 b^2\left(\frac{\psi_1(0)}{\psi_1(1)}\right)^{\frac{2n-2\ell-1}{4n}}}\right|^{\frac{1}{2}},
\tag{9}
$$

where

$$
\mathcal{M}_1 = \mathcal{V}(\omega_1^{k_1},\ldots,\omega_1^{k_{n-1}},\omega_1^\ell)\cdot\mathcal{V}(\omega_1^{2n-\ell-1},\omega_1^{k'_1},\ldots,\omega_1^{k'_{n-1}}),
$$

$$
\mathcal{M}_2 = \mathcal{V}(\omega_1^{k_1},\ldots,\omega_1^{k_{n-1}},\omega_1^{2n-\ell-1})\cdot\mathcal{V}(\omega_1^\ell,\omega_1^{k'_1},\ldots,\omega_1^{k'_{n-1}}).
$$

---

[1] Note that the normalization condition implies that the numbers k_ν and k'_ν, $\nu = 1,\ldots,n-1$, differ from ℓ and $2n-\ell-1$.

Proof. We have

$$\theta_{-1}(\psi) = (\psi(0))^{\frac{\varkappa_0}{2n} - \frac{(n-1)(2n-1)}{4n}} (\psi(1))^{\frac{\varkappa_1}{2n} - \frac{(n-1)(2n-1)}{4n}} \times$$

$$\times \det \begin{pmatrix} 1 & \omega_1^{k_1} & \cdots & \omega_{n-1}^{k_1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_1^{k_{n-1}} & \cdots & \omega_{n-1}^{k_{n-1}} & 0 & 0 & \cdots & 0 \\ \widetilde{\alpha}_n & \widetilde{\alpha}_n\omega_1^\ell & \cdots & \widetilde{\alpha}_n\omega_{n-1}^\ell & \widetilde{\gamma}_n\omega_n^\ell & \widetilde{\gamma}_n\omega_{n+1}^\ell & \cdots & \widetilde{\gamma}_n\omega_{2n-1}^\ell \\ \widetilde{\alpha}_{n+1} & \widetilde{\alpha}_{n+1}\omega_1^{2n-2\ell-1} & \cdots & \widetilde{\alpha}_{n+1}\omega_{n-1}^{2n-\ell-1} & \widetilde{\gamma}_{n+1}\omega_n^{2n-\ell-1} & \widetilde{\gamma}_{n+1}\omega_{n+1}^{2n-\ell-1} & \cdots & \widetilde{\gamma}_{n+1}\omega_{2n-1}^{2n-\ell-1} \\ 0 & 0 & \cdots & 0 & \omega_n^{k_1'} & \omega_{n+1}^{k_1'} & \cdots & \omega_{2n-1}^{k_1'} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \omega_n^{k_{n-1}'} & \omega_{n+1}^{k_{n-1}'} & \cdots & \omega_{2n-1}^{k_{n-1}'} \end{pmatrix} =$$

$$= (\psi(0))^{\frac{\varkappa_0}{2n} - \frac{(n-1)(2n-1)}{4n}} (\psi(1))^{\frac{\varkappa_1}{2n} - \frac{(n-1)(2n-1)}{4n}} \cdot (-1)^{\varkappa_1 + 2n - \ell - 1} \times$$

$$\times \left[\widetilde{\alpha}_n\widetilde{\gamma}_{n+1}\mathcal{V}(\omega_1^{k_1}, \ldots, \omega_1^{k_{n-1}}, \omega_1^\ell) \cdot \mathcal{V}(\omega_1^{2n-\ell-1}, \omega_1^{k_1'}, \ldots, \omega_1^{k_{n-1}'}) + \right.$$

$$\left. + \widetilde{\alpha}_{n+1}\widetilde{\gamma}_n\mathcal{V}(\omega_1^{k_1}, \ldots, \omega_1^{k_{n-1}}, \omega_1^{2n-\ell-1}) \cdot \mathcal{V}(\omega_1^\ell, \omega_1^{k_1'}, \ldots, \omega_1^{k_{n-1}'}) \right].$$

Since

$$\widetilde{\alpha}_n = a(\psi(0))^{\frac{2\ell+1-2n}{4n}}, \quad \widetilde{\alpha}_{n+1} = b(\psi(0))^{\frac{2n-2\ell-1}{4n}},$$

$$\widetilde{\gamma}_n = b(\psi(1))^{\frac{2\ell+1-2n}{4n}}, \quad \widetilde{\gamma}_{n+1} = a(\psi(1))^{\frac{2n-2\ell-1}{4n}},$$

we have

$$|\theta_{-1}(\psi)| = (\psi(0))^{\frac{\varkappa_0}{2n} - \frac{(n-1)(2n-1)}{4n}} (\psi(1))^{\frac{\varkappa_1}{2n} - \frac{(n-1)(2n-1)}{4n}} \times$$

$$\times \left| \mathcal{M}_1 a^2 \left(\frac{\psi(1)}{\psi(0)} \right)^{\frac{2n-2\ell-1}{4n}} + \mathcal{M}_2 b^2 \left(\frac{\psi(0)}{\psi(1)} \right)^{\frac{2n-2\ell-1}{4n}} \right|.$$

Now Corollary 1 implies (9). $\square$

The following relations can be obtained from Theorem 3 using [6, Theorem 3] and [19, Theorem 3.4].

Proposition 7. *Let $Y(t)$ be the Bogolyubov process (see [7, 8]). Then, under assumptions of Proposition 1, the following relation holds:*

$$\mathsf{P}\{\|Y_m^{[\beta_1,\ldots,\beta_m]}\|_\psi \le \varepsilon\} \sim \left(\frac{\psi(0)}{\psi(1)} \right)^{\frac{m(m+2)}{8(m+1)} - \frac{\mathcal{K}}{4(m+1)}} \times$$

$$\times \left| \prod_{\nu=1}^m |1 + z_{m+1}^{k_\nu}|^2 \left(\frac{\psi(0)}{\psi(1)} \right)^{\frac{1}{4(m+1)}} + \prod_{\nu=1}^m |1 + z_{m+1}^{2m+1-k_\nu}|^2 \left(\frac{\psi(1)}{\psi(0)} \right)^{\frac{1}{4(m+1)}} \right|^{-\frac{1}{2}} \times$$

$$\times \frac{2^{m+2}(m+1)^{m+1}\sinh(\omega/2)}{|\mathcal{V}(z_{m+1}^{k_1}, \ldots, z_{m+1}^{k_m})|} \frac{\varepsilon_{m+1}}{\sqrt{\pi\mathcal{D}_{m+1}}} \exp\left(-\frac{\mathcal{D}_{m+1}}{2\varepsilon_{m+1}^2} \right),$$

where $k_\nu = \nu - (2\nu + 1)\beta_\nu$, $\nu = 1, \ldots, m$.

Consider multiply centered-integrated Brownian bridge:

$$B_{\{0\}}(t) = B(t), \quad B_{\{l\}}(t) = \int_0^t \overline{B_{\{l-1\}}}(s)ds, \quad l \in \mathbb{N}.$$

Proposition 8. *Suppose that the function $\psi \in W_\infty^{m+2}(0,1)$ is bounded away from zero and satisfies the relation $\int_0^1 \psi^{\frac{1}{2(m+2)}}(x)dx = 1$. Then the following relation holds:*

$$\mathsf{P}\{\|(B_{\{1\}})_m^{[\beta_1,\ldots,\beta_m]}\|_\psi \leq \varepsilon\} \sim (\psi(0))^{\frac{m^2-3}{8(m+2)}-\frac{\widetilde{\mathcal{K}}}{4(m+2)}} (\psi(1))^{\frac{\widetilde{\mathcal{K}}}{4(m+2)}-\frac{m^2+8m+3}{8(m+2)}} \times$$

$$\times \left| \prod_{\nu=1}^m \left|1+z_{m+2}^{k_\nu}\right|^2 \left(\frac{\psi(0)}{\psi(1)}\right)^{\frac{1}{4(m+2)}} + \prod_{\nu=1}^m \left|1+z_{m+2}^{2m+3-k_\nu}\right|^2 \left(\frac{\psi(1)}{\psi(0)}\right)^{\frac{1}{4(m+2)}} \right|^{-\frac{1}{2}} \times$$

$$\times \frac{(2m+4)^{\frac{m+2}{2}}\sqrt{2\sin\frac{3\pi}{2m+4}}}{|\mathcal{V}(z_{m+2}^{k_1},\ldots,z_{m+2}^{k_m})|} \frac{\varepsilon_{m+2}^{-2}}{\sqrt{\pi\mathcal{D}_{m+2}}} \exp\left(-\frac{\mathcal{D}_{m+2}}{2\varepsilon_{m+2}^2}\right),$$

where $\widetilde{\mathcal{K}} = \widetilde{\mathcal{K}}(\beta_1,\ldots,\beta_m) = \sum_{\nu=1}^m (2\nu+3)\beta_\nu$ and $k_\nu = \nu - (2\nu+3)\beta_\nu$, $\nu = 1,\ldots,m$.

Now we consider the case of boundary conditions periodic in the main terms. The following theorem can be easily derived from Corollary 1.

Theorem 4. *Let the assumptions of Corollary 1 be satisfied. Suppose also that the boundary conditions (2) have the form*

$$v^{(\nu)}(0) - v^{(\nu)}(1) + \sum_{j=0}^{\nu-1}\left(\alpha_{\nu j}v^{(j)}(0) + \gamma_{\nu j}v^{(j)}(1)\right) = 0, \quad \nu = 0,\ldots,2n-1.$$

Then

$$\lim_{\varepsilon\to 0}\frac{\mathsf{P}(\|X\|_{\psi_1}\leq\varepsilon)}{\mathsf{P}(\|X\|_{\psi_2}\leq\varepsilon)} = \left(\frac{\psi_1(0)\psi_1(1)}{\psi_2(0)\psi_2(1)}\right)^{\frac{2n-1}{8}} \times$$

$$\times \left|\frac{\mathcal{V}((\psi_2(0))^{\frac{1}{2n}},(\psi_2(0))^{\frac{1}{2n}}\omega_1,\ldots,(\psi_2(0))^{\frac{1}{2n}}\omega_{n-1},(\psi_2(1))^{\frac{1}{2n}}\omega_n,\ldots,(\psi_2(1))^{\frac{1}{2n}}\omega_{2n-1})}{\mathcal{V}((\psi_1(0))^{\frac{1}{2n}},(\psi_1(0))^{\frac{1}{2n}}\omega_1,\ldots,(\psi_1(0))^{\frac{1}{2n}}\omega_{n-1},(\psi_1(1))^{\frac{1}{2n}}\omega_n,\ldots,(\psi_1(1))^{\frac{1}{2n}}\omega_{2n-1})}\right|^{\frac{1}{2}}.$$

The following relation can be obtained from Theorem 4 using [19, Theorem 3.2].

Proposition 9. *Under assumptions of Proposition 1, the following relation holds:*

$$\mathsf{P}(\|\overline{B_{\{m\}}}\|_\psi \leq \varepsilon) \sim (\psi(0)\psi(1))^{\frac{2m+1}{8}} \times$$

$$\times \left|\mathcal{V}((\psi(0))^{\frac{1}{2(m+1)}},(\psi(0))^{\frac{1}{2(m+1)}}z_{m+1},\ldots,(\psi(0))^{\frac{1}{2(m+1)}}z_{m+1}^m,(\psi(1))^{\frac{1}{2(m+1)}}z_{m+1}^{m+1},\ldots,(\psi(1))^{\frac{1}{2(m+1)}}z_{m+1}^{2m+1})\right|^{-\frac{1}{2}} \times$$

$$\times (2m+2)^{\frac{m+2}{2}} \frac{\varepsilon_{m+1}^{-(2m+1)}}{\sqrt{\pi\mathcal{D}_{m+1}}} \exp\left(-\frac{\mathcal{D}_{m+1}}{2\varepsilon_{m+1}^2}\right).$$

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